Approximate ESPs in Simple Cubic Polytopes Using a Rubberband Algorithm

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Abstract

Let p and q be two points inside a special type of cubic polytope Π. This paper provides a rubberband algorithm for computing a Euclidean shortest path between p and q (3D ESP) that is contained inside Π. The algorithm has $\kappa_1(\varepsilon) \cdot \kappa_2(\varepsilon) \cdot \mathcal{O}(n^2)$ time complexity, where n is the number of cubes consists of Π, $\kappa_i(\varepsilon) = (L_{0i} - L_i)/\varepsilon$, for the true length L_i of some shortest path with initial (polygonal path) length L_{0i} (used when approximating this shortest path), for i = 1, 2. Rubberband algorithms follow a straightforward design strategy, and the proposed algorithm is easy to implement and thus of importance for applications.

The given rubberband algorithm can also be applied to solve approximately three NP-complete or NP-hard 3D Euclidean shortest path (ESP) problems in time $\kappa(\varepsilon) \times \mathcal{O}(k)$, where k is the number of layers in a stack which contains the defined obstacles.

1. Introduction

LET Π be a connected polyhedral domain such that its frontier is a union of a finite number of triangles. An *obstacle* is a connected, bounded polyhedral component in the complement $\mathbb{R}^3 \backslash \Pi$ of Π. Let $p, q \in \Pi$ such that $p \neq q$. The *general Euclidean shortest-path problem* (ESP) asks to find the shortest polygonal path $\rho(p, q)$ which is either completely contained in Π, or just not intersecting any (topologic) interior of a finite number of given obstacles.

This problem is actually a special case of the problem of planning optimal collision-free paths for a robot system; for its specification and a first result, see [13]. This paper presented in 1984 a doubly exponential time algorithm for solving the general obstacle avoidance problem. [12] improved this by providing a singly exponential time algorithm. The result was further improved by a PSPACE algorithm in [3]. Since the general ESP problem is known to be NP-hard [2], special cases of the problem have been studied afterwards. [14] gave a polynomial time algorithm for ESP calculations for cases where all obstacles are convex and the number of obstacles is small. [5] solved the ESP problem with an $\mathcal{O}(n^{6k-1})$ algorithm assuming that all obstacles are vertical "buildings" with k different values for height.

[13] is the first publication considering the special case that the shortest polygonal path $\rho(p, q)$ is constrained to stay on the surface of Π. [13] presented an $\mathcal{O}(n^3 \log n)$ algorithm where Π was assumed to be convex. [10] improved this result by providing an $\mathcal{O}(n^2 \log n)$ algorithm for the surface of any bounded polyhedral Π. The time complexity was even reduced to $\mathcal{O}(n^2)$ [4]. So far, the best known result for the surface ESP problem is due to [6]; it improved in 1999 the time complexity to $\mathcal{O}(n \log^2 n)$, assuming that there are $\mathcal{O}(n)$ vertices and edges on Π.

This paper provides a rubberband algorithm (RBA) for computing approximate ESP inside a special type of cubic polytopes. The algorithm has $\kappa_1(\varepsilon) \cdot \kappa_2(\varepsilon) \cdot \mathcal{O}(n^2)$ time complexity, where n is the number of cubes made of Π, $\kappa_i(\varepsilon) = (L_{0i} - L_i)/\varepsilon$, for the true length L_i of some kind of shortest path with length L_{0i} of the used initial polygonal path, for i = 1, 2. This rubberband algorithm follows a straightforward design strategy, and the proposed algorithm is easy to implement. We generalize a rubberband algorithm from solving the 2D ESP of a simple polygon (see [8]) to a solution for the surface ESP of polytopes (see [9]), and then to special type of cubic polytopes. Considering the difficulty of the general ESP problem, our approach is very important for exploring efficient approximate algorithms for general 3D ESP problem.

2. Definitions

A *cubic* polytope, denoted by Π, is a union of finite number of cubes such that Π is homeomorphic to a polytope. Assume that polytope Π is placed in a Cartesian xyz-coordinate system. Then there exist three ways (i.e., parallel to plane xOy, yOz or zOx) to decompose Π into some cubic layers. If Π can be decomposed into some layers such that the intersection set of any two consecutive layers is a set of regions of disjointed simple polygons and the region graph(i.e., a simple graph $G = [V, E]$, where V is the set of all regions, and E is the set of all unordered regions which belong to a common layer.) is a tree, then Π is called a *simple cubic polytope*. The frontier of each region is called a *region polygon*. If a simple polygon contains at least two region polygons and each edge of it is the union of the edges of some cubes of a layer, then it is called a *stable polygon* (see the red color polygon in Figure 2.).

For example, Figure 1 shows a simple cubic polytope. In Figure 2, the 6 blue color simple polygons are the region polygons of the simple cubic polytope shown in Figure 1.

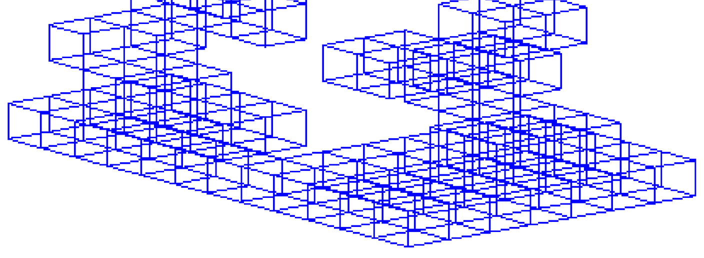

Figure 1: *A simple cubic polytope.*

Figure 2: *Each blue color polygon is a region polygon of the simple cubic polytope shown in Figure 1; the red color polygon (i.e., the largest one) is a stable polygon.*

3. The Main Steps of the Algorithm

1. Find a sequence of region polygons and stable polygons such that there is a initial path ρ from p to q such that ρ is completely contained in Π and each vertex of ρ is located on the frontier of a region polygon or contained inside a stable polygon.
2. Decompose each stable polygon into rectangles to find a sequence of segments such that each path from p to q must pass through these segments.
3. From Steps 1 and 2, find a sequence (i.e., step set) of region polygons or segments such that each path from p to q must pass through each element of the sequence (each element is called a *step element*).
4. Apply a rubberband algorithm (see Section 8.3.1 of [7]) on the step set to compute an approximate ESP from p to q.

4. An Example

Let p and q be two points inside a simple cubic polytope Π shown in Figure 1 (i.e., the two endpoints of the green polygonal path (i.e., an initial path) shown in Figure 2.).

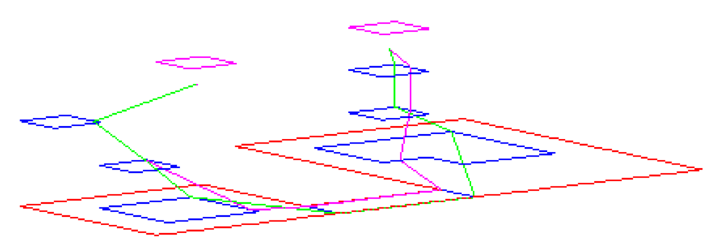

Figure 3: *The pink color path is the resulting path after the first iteration of the algorithm.*

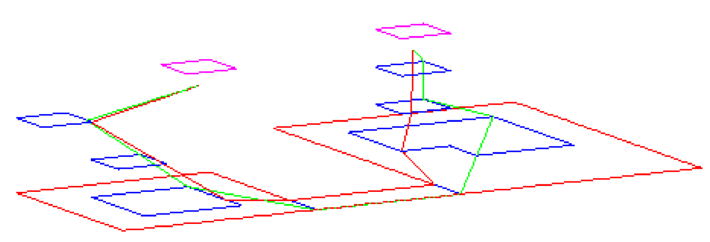

Figure 4: *The red color path is the final path (after the 9th iteration with the accuracy 10^{-10}) obtained from the algorithm.*

References

[1] J. Canny and J. H. Reif. New lower bound techniques for robot motion planning problems. In Proc. *IEEE Conf. Foundations Computer Science*, pages 49–60, 1987.

[2] J. Canny and J. H. Reif. New lower bound techniques for robot motion planning problems. In Proc. *IEEE Conf. Foundations Computer Science*, pages 49–60, 1987.

[3] J. Canny. Some algebraic and geometric configurations in PSPACE. In Proc. *ACM Symp. Theory Computation*, pages 460–467, 1988.

[4] J. Chen and Y. Han. Shortest paths on a polyhedron. In Proc. *ACM Symp. Computational Geometry*, pages 360–369, 1990.

[5] L. P. Gewali, S. Ntafos, and I. G. Tollis. Path planning in the presence of vertical obstacles. Technical Report, Computer Science, University of Texas at Dallas, 1989.

[6] S. Kapoor. Efficient computation of geodesic shortest paths. In Proc. *ACM Symp. Theory Computation*, pages, 770–779, 1999.

[7] F. Li and R. Klette. Exact and approximate algorithms for the calculation of shortest paths. Report 2141, IMA, The University of Minnesota, Minneapolis, 2006.

[8] F. Li and R. Klette. Euclidean shortest paths in simple polygons. Technical Report CITR-202, Computer Science Department, The University of Auckland, Auckland, 2007.

[9] F. Li and R. Klette. Approximate ESPs on Surfaces of Polytopes Using a Rubberband Algorithm. In Proc. *PSIVT 2007*, LNCS, to appear.

[10] J. S. B. Mitchell, D. M. Mount, and C. H. Papadimitriou. The discrete geodesic problem. *SIAM J. Computation*, **16**:647–668, 1987.

[11] J. S. B. Mitchell. Geometric shortest paths and network optimization. In *Handbook of Computational Geometry* (J.-R. Sack and J. Urrutia, editors). pages 633–701, Elsevier, 2000.

[12] J. H. Reif and J. A. Storer. A single-exponential upper bound for shortest paths in three dimensions. *J. ACM*, **41**:1013–1019, 1994.

[13] M. Sharir and A. Schorr. On shortest paths in polyhedral spaces. *SIAM J. Computation*, **15**:193–215, 1986.

[14] M. Sharir. On shortest paths amidst convex polyhedra. *SIAM J. Computation*, **16**:561–572, 1987.